Final Examination, 9 MAY 1997
SM311O (Spring 1997)

The following formulas may be useful to you:

$$a)\ \int\int_S \mathbf{v}\cdot d\mathbf{A} = \int\int\int_D \text{div}\ \mathbf{v}\, dxdydz, \qquad b)\ \oint_C \mathbf{v}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{v}\cdot d\mathbf{A},$$

$$c)\ \rho(\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\cdot\mathbf{v}) = -\nabla p + \mu\Delta\mathbf{v} + \rho\mathbf{F}, \qquad \text{div}\ \mathbf{v} = 0.$$

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Part 1

1. Find the solution to the initial value problem

$$x' = x + y, \quad y' = -x + y, \quad x(0) = 0, y(0) = -2.$$

2. Find the solution to the initial-boundary value problem

$$u_t = 4u_{xx}, \quad u(0,t) = u(\pi, t) = 0, \quad u(x,0) = 3\sin x.$$

3. (a) Let f be a function of two variables. Describe the geometric relationship between the gradient and the contours of f.

   (b) Let $T(x,y) = x^2 + y^2 - 2x$ be the temperature profile of a two-dimensional body of water, with x and y the coordinates of a typical fluid particle. Draw the graph of the 1-isotherm, i.e., the set of all points that have temperature equal to 1.

   (c) Let $\mathbf{v}$ be a two-dimensional vector field. Define mathematically what it means for $\mathbf{v}$ to have a potential and a stream function. State the necessary conditions (in terms of vector operations) for $\mathbf{v}$ to have a potential and a stream function.

   (d) Let $\mathbf{v}$ be a two-dimensional vector field with a potential ϕ and a stream function ψ. Show that the contours of ϕ and ψ must be orthogonal to each other.

4. Let $\mathbf{v} = \langle 4xy^3 - y, -x + 6x^2y^2, 6z\rangle$.

   (a) Does $\mathbf{v}$ have a potential ϕ? If no, explain. If yes, find it.

   (b) Compute $\int_C \mathbf{v}\cdot d\mathbf{r}$ where C is the straight line connecting $(0,0,0)$ with $(1,-1,2)$.

5. Let $\mathbf{v} = \langle x-2y, 3x-y\rangle$. Show that this vector field has a stream function and proceed to determine it. Apply this stream function to determine the equation for the path traversed by the particle located at position $(1,-2)$ at time 0.

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Part 2

6. (a) Write down a parametrization $\mathbf{r}(u, v)$ of S if
    i. S is the plane that passes through the points $(1, -1, 3)$, $(1, 1, 2)$ and $(0, 0, 0)$.
    ii. S is a sphere of radius 3 centered at $(2, -1, 2)$.

   (b) Find a unit normal vector to the surface $z = 3x^2 + 4y^2$ at $P = (1, 2)$.

7. Let $\mathbf{v}(x, y, z) = \langle 0, 0, 2z - 1\rangle$ be the velocity field of a fluid flow. Find the flux of this flow through the set of points on the surface $z = 1 - x^2 - y^2$ and located in the upper-half space $z > 0$.

8. Use double or triple integrals to compute the volume of the tetrahedron with vertices, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(1, 1, 3)$.

9. (a) Let $\psi(x, y) = \cosh \pi x \cos \pi y - 2 \sinh \pi x \sin \pi y$ be the stream function of a fluid flow. Find the velocity at $(x, y) = (1, 1)$.

   (b) Let $\mathbf{v} = \langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \rangle$. Find the vorticity of $\mathbf{v}$ at $(x, y) = (1, 1)$.

10. Let $\mathbf{v}(x, y, z) = \langle 3x^2, -y^2, 0\rangle$ be the velocity field of a fluid whose density and viscosity are equal to unity. The position of a fluid particle is denoted by (x, y, z).

    (a) Find the acceleration of the particle that occupies $(1, -1, 1)$.

    (b) Verify whether there is a pressure function p such that the pair $(\mathbf{v}, p)$ satisfies the Navier-Stokes equations with the body force $\mathbf{F} = \mathbf{0}$. If these equations are satisfied, what is the associated pressure p?